On the Effect of the Form of the Transverse Section on the Frictional Resistance to the Motion of an Elongated Body Parallel to its Length through a Fluid.

By Charles H. Lees, D.Sc., F.R.S.

(Received September 28, 1915.)

1. The great increase in the lengths of the parallel mid-bodies of recently constructed submarines and airships has raised into prominence the question of the frictional resistance of such elongated bodies moving parallel to their length through fluids, like air and water, whose viscosities cannot be neglected. This resistance increases as the length increases, and becomes comparable with the head and tail resistances, which for short bodies constitute nearly the whole resistance.

In general, the problem of greatest practical importance is the determination of the frictional resistance when the motion is rapid enough to produce eddy currents in the fluid, but the difficulties in the way of a general theory of eddy current motion have prevented a solution being reached. The simpler problem of the resistance offered by the walls of a circular pipe to the turbulent flow of viscous fluid through the pipe has formed the subject of extensive series of experiments by Saph and Schoder,* of Cornell University, and by Stanton and Pannell† at the National Physical Laboratory.

As a result of these observations, it has been shown that the fall of pressure per unit length along a smooth pipe of diameter d, through which a fluid of density ρ and kinematic viscosity ν is flowing with mean velocity v, is equal to $\{\rho v^2/d\}\,\{a+b(\nu/vd)^n\}$, where a, b, and n are constants, the values of which are the same over a wide range of fluids and diameters of pipes, viz., $a = 0{\cdot}0036$, $b = 0{\cdot}306$, and $n = 0{\cdot}35$.‡

2. So far as I know, no observations on the resistance of a pipe with a central solid core concentric with the inner wall of the pipe, have been made with a fluid in turbulent motion through the annulus. But in the case of

---

* Saph and Schoder, 'Amer. Soc. Civil Engrs. Proc.,' vol. 51, p. 253 (1903).

† Stanton and Pannell, 'Phil. Trans.,' A, vol. 214, p. 199 (1914).

‡ Lees, 'Roy. Soc. Proc.,' A, vol. 91, p. 46 (1914). I take this opportunity of correcting several numerical errors in the above paper. In the six equations at the foot of p. 49, the three equations at the foot of p. 50, and the equation at the foot of p. 52, the quantity l should read $2l$; p. 48, last line, and p. 49, line 1, $\log(vl/\nu)$ should read $\log(vd/\nu)$; p. 49, line 17, 0·25 should read 0·208; p. 49, line 20, 0·0191 should read 0·0159; p. 49, line 3 from below, 0·0382 should read 0·0318.

stream-line motion through the pipe, it can readily be shown that the resistance per unit area is not the same at the inner and outer surfaces. If the circular pipe is of internal radius b, with a concentric core of radius a, the velocity of flow v at a point r from the common axis, due to a pressure slope along the pipe of p per centimetre, is given by

$$v = (p/4\eta)\{-(r^2-a^2)+(b^2-a^2)\log(r/a)/\log(b/a)\},$$

where η is the viscosity of the fluid.

The total flow per second through the annular section is V c.c., where

$$\mathrm{V} = \{\pi(b^2-a^2)(p/8\eta)\}\{b^2+a^2-(b^2-a^2)/\log(b/a)\}.^*$$

From these expressions it follows that the resistances per unit area R_a and R_b of the inner and outer walls respectively are given by

$$\mathrm{R}_a = \frac{\mathrm{V}}{\pi(b^2-a^2)}\frac{2\eta}{a}\left(\frac{1}{(b^2+a^2)/(b^2-a^2)-1/\log(b/a)}-1\right)$$

$$\mathrm{R}_b = \frac{\mathrm{V}}{\pi(b^2-a^2)}\frac{2\eta}{b}\left(\frac{1}{(b^2+a^2)/(b^2-a^2)-1/\log(b/a)}+1\right).$$

The first term on the right is the mean velocity over the cross-section, the second varies inversely as the radius of the surface considered, while the third varies more slowly with the ratio of the radii of the two surfaces.

Whatever be the ratio of the radii of the two surfaces, the resistance per unit area is greater for the surface of small than for that of large radius, and it seems advisable to test by experiment whether this is also the case when the motion is turbulent instead of stream-line flow.

If F_a and F_b are the forces exerted on the inner and outer surfaces respectively per unit length we have

$$\mathrm{F}_a = 2\pi a\mathrm{R}_a \qquad \text{and} \qquad \mathrm{F}_b = 2\pi b\mathrm{R}_b.$$

Hence the ratio of the force exerted on the outer surface to the total force on both is given by

$$\frac{\mathrm{F}_b}{\mathrm{F}_b+\mathrm{F}_a} = \frac{b\mathrm{R}_b}{b\mathrm{R}_b+a\mathrm{R}_a} = \frac{1}{1-a^2/b^2} - \frac{1}{2\log_e(b/a)}.$$

In fig. 1 the full line shows how this ratio varies as a/b changes from 0 to 1. The dotted line shows how it would vary if the ratio of the total forces were equal to the ratio of the $(\text{radii})^{2/3}$ or the resistances per unit area inversely as the cube roots of the radii. For values of a/b between 0·4 and 1·0 the inverse cube root law is a very close approximation to the facts, while over the range from 0·2 to 1·0 it is not so seriously in error as to render it unsuitable for practical purposes.

---

* Compare Lamb, 'Hydrodynamics,' 3rd Edit., p. 545.

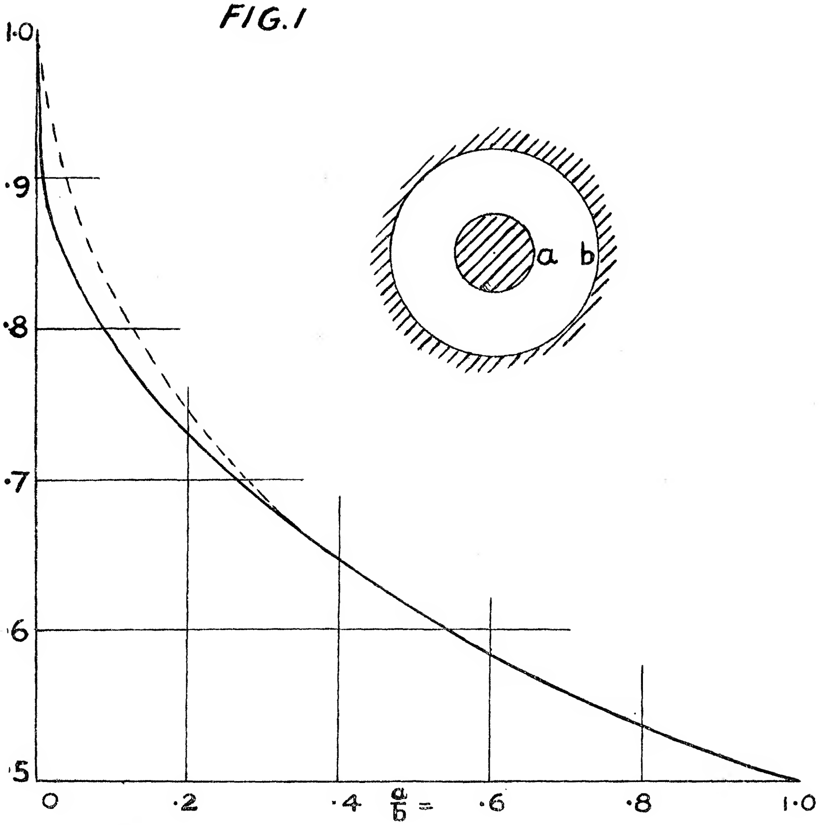


Fig. 1.—Full line—ratio of force on outer surface to total force on both. Dotted line—ratio if resistances of the two surfaces per unit area were inversely proportional to the cube roots of the radii of the surfaces.

3. In the case of stream-line flow through a pipe of elliptical section with a and b for semi-axes major and minor respectively, it has been shown by Greenhill* that the velocity v at the point (x, y) is given by

$$v = \frac{pa^2b^2}{2\eta(a^2+b^2)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),$$

where, as before, p is the slope of pressure along the pipe per cm. and η is the viscosity of the fluid.

The maximum velocity occurs at the centre and is $= pa^2b^2/2\eta(a^2+b^2)$.

The total delivery per second is V c.c., where

$$V = p\pi a^3b^3/4\eta(a^2+b^2),\dagger$$

---

* Greenhill, 'Lond. Math. Soc. Proc.,' vol. 13, p. 43 (1881). Results for triangular and rectangular sections are also given in this paper.

† Lamb, 'Hydrodynamics,' 3rd Edit., p. 546.

and the mean velocity $\bar{v} = V/\pi ab = pa^2b^2/4\eta(a^2+b^2)$, that is half the maximum.

If F is the force exerted on 1 cm. length of the wall of the pipe

$$F = \pi abp = 4\pi\eta\bar{v}(a^2+b^2)/ab$$

The resistance R per unit area of the wall at (x', y') is

$$\eta\, dv/dn = \eta\left(l\frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y}\right)$$

where n is measured along the normal at the point (x', y'), whose direction cosines are l and m.

FIG. 2

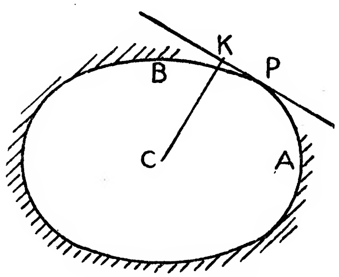


Fig. 2.—The resistance per unit area at P is inversely as CK.

Hence $$R = 4\eta\bar{v}\sqrt{(x'^2/a^4+y'^2/b^4)} = 4\eta\bar{v}/CK,$$

where CK is the perpendicular from the centre of the ellipse on to the tangent at (x', y'). Thus the resistance per unit area of the surface of the pipe at any point is inversely as the length of the perpendicular from the centre of the ellipse on the tangent at the point. The resistances per unit area at the ends of the axes a, b are as $1/a : 1/b$ respectively.

The mean resistance per unit area over the whole contour of the section is

$$\bar{R} = \pi\eta\bar{v}(a^2+b^2)/a^2bE_1(\alpha),$$

where $\cos\alpha = b/a$ and $E_1(\alpha)$ is the complete second elliptic integral.

For $aE_1(\alpha)$, the length of a quadrant of the ellipse, we may in many cases substitute the approximate expression $\frac{1}{4}\pi\{\frac{1}{2}(a+b)+\sqrt{[\frac{1}{2}(a^2+b^2)]}\}$, and we then get for the mean resistance

$$\bar{R} = 4\pi\bar{v}\frac{a^2+b^2}{ab\{\frac{1}{2}(a+b)+\sqrt{[\frac{1}{2}(a^2+b^2)]}\}}.$$

This reduces for a circular pipe to $\bar{R} = 4\eta\bar{v}/a$, a well-known equation.

It will be noticed that the surfaces of equal velocity are elliptic cylinders similar to and concentric with the inner surface of the pipe, and it is a matter of importance to ascertain whether this is also the case when the flow becomes turbulent.

4. From the remarks in Sections 1 and 2, it will be gathered that our knowledge of the frictional resistance of an elementary area of a solid along which a fluid is in turbulent motion is very limited, except in the case of flow through circular pipes, and is in the main empirical. For the frictional resistance of bodies immersed in the fluid stream, we are dependent chiefly on Froude's* experiments on flat boards towed through water and on Zahm's† on boards placed in a current of air.

In both cases, for want of knowledge of the relative effects of flat surfaces, of edges and of corners, the mean frictional resistance only could be ascertained, the total frictional resistance found being divided by the total area in contact with the fluid. These mean frictional resistances have proved sufficiently accurate for practical purposes for many years, but the necessity for further investigation has been felt for some time by naval architects and others.‡

In view of this demand for further information, it is proposed in the following sections to examine the frictional resistances of a few very elongated bodies moving parallel to their long axes, with speeds not high enough to produce turbulent motion in the fluids through which they move, and to determine the effects of the shape of the section of the bodies and the influence of projecting keels or ridges on that resistance.

In order to keep the head and tail resistances out of the problems, the bodies are taken as infinitely long, and are supposed to move parallel to their lengths along a concentric tube or pipe whose diameter is supposed to be large.

5. *Case* 1.—A long elliptic cylinder is in motion parallel to its length through the fluid in a wide tube whose inner contour is a confocal ellipse. To determine the resistance to the motion of the cylinder, the fluid being assumed in stream-line motion. Take the centre of the ellipses as origin of co-ordinates, and let the x axis pass through the foci, each being at a distance a_0 from the centre, and let $z = x+iy$. The confocal ellipses are converted into lines parallel to the u axis in the w plane, where $w = u+iv$ by the transformation

$$z/a_0 = \sin(w/\alpha), \qquad \text{where } \alpha \text{ is a constant.}$$

---

* Froude, 'Brit. Assoc. Rep.,' p. 118 (1872), and p. 249 (1873).

† Zahm, 'Phil. Mag.,' vol. 8, p. 58 (1904).

‡ See, for example, Sir W. White's remarks in 'Trans. Naval Arch.,' vol. 46, p. 46 (1904).

This gives, on expanding and separating real and unreal parts,

$$x = a_0 \sin(u/\alpha)\cosh(v/\alpha) \qquad y = a_0 \cos(u/\alpha)\sinh(v/\alpha).$$

Hence

$$\frac{x^2}{a_0{}^2 \cosh^2(v/\alpha)} + \frac{y^2}{a_0{}^2 \sinh^2(v/\alpha)} = 1$$

is the equation to the ellipses of equal velocity, and

$$\frac{x^2}{a_0{}^2 \sin^2(u/\alpha)} - \frac{y^2}{a_0{}^2 \cos^2(u/\alpha)} = 1$$

the equation to the lines of flow of the viscous forces; these lines are a series of confocal hyperbolas.

The semi-axes major and minor of the ellipses of constant velocity are $a_0 \cosh(v/\alpha)$ and $a_0 \sinh(v/\alpha)$ respectively, and the sum of the two is $a_0 \exp(v/\alpha)$. Hence, if v_1 is the velocity at the ellipse of semi-axes a_1 and b_1, and v_2 that at the confocal ellipse of semi-axes a_2 and b_2, so that $a_2{}^2 - b_2{}^2 = a_1{}^2 - b_1{}^2 = a_0{}^2$ (fig. 3), we have

$$\alpha = \frac{v_2 - v_1}{\log[(a_2 + b_2)/(a_1 + b_1)]}.$$

Since the complete period of u is $2\pi\alpha$, and the force transmitted per period is $2\pi\alpha\eta$, the viscous force F transmitted outwards through the whole surface of unit length of each confocal cylinder is

$$\mathrm{F} = \frac{2\pi\eta(v_1 - v_2)}{\log[(a_2 + b_2)/(a_1 + b_1)]}.$$

Thus the resistance to the motion of an elliptic cylinder in a confocal

FIG. 3

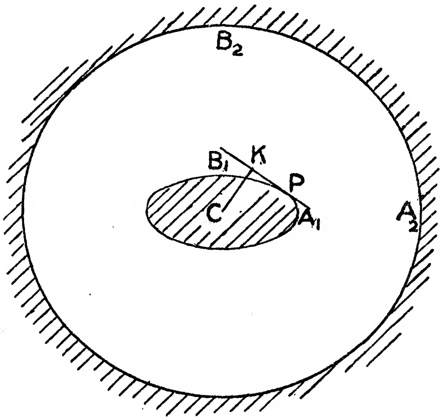


Fig. 3.—Elliptic cylinder moving along confocal elliptic tube. The resistance per unit area at P is proportional to CK.

elliptic tube is the same as that of a circular cylinder in a circular tube of radii equal respectively to the means of the semi-axes of the ellipses.

The resistance per unit area at the point (x', y') of the ellipse of semi-axes a, b, is given by $\eta\, dv/dn$, that is, by $\eta\alpha\, \mathrm{CK}_1/a_1b_1$, where

$$\mathrm{CK}_1 = 1/\sqrt{(x'^2/a_1^4 + y'^2/b_1^4)}$$

is the length of the perpendicular from the centre of the ellipse on to the tangent at (x', y').

Hence $$\mathrm{R}_1 = \eta\,\frac{\mathrm{CK}_1}{a_1b_1}\,\frac{v_1-v_2}{\log\left[(a_2+b_2)/(a_1+b_1)\right]},\text{ for the inner}$$

and $$\mathrm{R}_2 = \eta\,\frac{\mathrm{CK}_2}{a_2b_2}\,\frac{v_1-v_2}{\log\left[(a_2+b_2)/(a_1+b_1)\right]},\text{ for the outer cylinder.}$$

The mean resistance $\overline{\mathrm{R}}_1$ per unit area for the inner cylinder is given by

$$\overline{\mathrm{R}}_1 = \eta\,\frac{2\pi}{a_1 4\mathrm{E}_1(\alpha_1)}\,\frac{v_1-v_2}{\log\left[(a_2+b_2)/(a_1+b_1)\right]},$$

where $\cos\alpha_1 = b_1/a_1$ and $\mathrm{E}_1(\alpha_1)$ is the complete second elliptic integral, and for the outer surface by

$$\overline{\mathrm{R}}_2 = \eta\,\frac{2\pi}{a_2 4\mathrm{E}_1(\alpha_2)}\,\frac{v_1-v_2}{\log\left[(a_2+b_2)/(a_1+b_1)\right]}.$$

In each case $a\mathrm{E}_1(\alpha)$ may for most purposes be replaced by

$$\tfrac{1}{4}\pi\left\{\tfrac{1}{2}(a+b)+\sqrt{\left[\tfrac{1}{2}(a^2+b^2)\right]}\right\}.$$

It will be noticed that the mean resistances of inner and outer surfaces are inversely as the areas of those surfaces, which it is evident from other considerations should be the case.

From the expressions for the resistances per unit area R_1 and R_2 at different parts of the elliptic surfaces it is also evident that the resistances of equal small areas near the ends of the major and minor axes of either surface are inversely proportional to the lengths of the minor and major axes respectively of that surface.

If the confocal ellipses become concentric circles owing to a_0 vanishing, the total viscous force transmitted per unit length is $\mathrm{F} = \dfrac{2\pi\eta(v_1-v_2)}{\log(r_2/r_1)}$ and the expressions for the resistance per unit area reduce for the inner cylinder to

$$\mathrm{R}_1 = \frac{\eta(v_1-v_2)}{r_1\log(r_2/r_1)}$$

and for the outer cylinder to

$$\mathrm{R}_2 = \frac{\eta(v_1-v_2)}{r_2\log(r_2/r_1)}$$

which are well known.

If on the other hand the inner ellipse reduces to the straight line of length $2a_0$ joining the foci, and the velocity of the plane it represents is v_0, the total resistance of the plane reduces to

$$\mathrm{F} = \frac{2\pi\eta(v_0 - v_2)}{\log[(a_2 + b_2)/a_0]}$$

and the resistance per unit area at the point x, to

$$\mathrm{R} = \eta \frac{1}{\sqrt{(a_0^2 - x^2)}} \frac{v_0 - v_2}{\log[(a_2 + b_2)/a_0]}.$$

This expression shows how much greater is the resistance per unit area near the edges of the strip than near its centre line.

A comparison of the expressions for the total resistances of a circular rod moving in a wide concentric pipe and of a flat strip moving in a wide confocal pipe, the radius of the wide circular pipe being equal to the mean of the axes major and minor of the wide confocal pipe, shows that the total resistance of a strip of width $2a$ and of contour $4a$ in contact with the fluid is equal to that of a circular rod of diameter a and contour πa.

Hence the frictional coefficients deduced for surfaces moving in water by Froude,* and in air by Zahm,* from experiments on thin boards should, if the relations here established for stream-line flow hold when the flow becomes turbulent, be multiplied by $4/\pi$, to give the coefficients for circular cylinders under the same conditions.

If a strip of breadth $2a$ moves parallel to its length between infinite planes parallel to and at a distance b from the strip instead of in a confocal pipe, the resistance to the motion of the strip is the same as that of a strip of width $2a + (4b/\pi)\log_\epsilon 2$† forming part of an infinite plane substituted for the strip and moving with the same speed. This result again brings out the exceptional part contributed by the edges of strips to their total resistance.

6. *Case* 2.—A long rod of rectangular section is in motion parallel to its length through a viscous liquid contained in a wide pipe of nearly circular section: to determine the resistance to the motion of the rod.

It has been shown by Schwarz‡ that if $z = x + iy$ and $\zeta' = \xi' + i\eta'$ the transformation $z = \int_1^{\zeta'} \frac{\sqrt{(1+\zeta'^4)}}{\zeta'^2} d\zeta'$ converts the area outside the square in the z plane with sides $x = 0$ and $= -2$, $y = \pm 1$ into the interior of the circle

---

* *Loc. cit.*, p. 148.

† Maxwell, 'Electricity and Magnetism,' 2nd Edit., vol. 1, p. 278; or Thomson, 'Recent Researches,' p. 218.

‡ Schwarz, 'Ges. Werke, vol. 2, p. 77; and Forsyth, 'Theory of Functions of a Complex Variable,' 2nd Edit., p. 639.

$\xi'^2 + \eta'^2 = 1$ in the ζ' plane, the centre of the circle corresponding to the circle at ∞ in the z plane (fig. 4).

In order to convert the area outside the normal rectangular section of the rod in the z plane into the interior of the same circle in the ζ' plane it is evident we only require to write $1-2\cos 2\alpha\,.\,\zeta'^2+\zeta'^4$ instead of $1+\zeta'^4$ under the radical. For the sake of comparison of the results for the case of the rectangular rod with those for a circular rod, it is convenient to invert the ζ' circle with respect to its own centre by writing $1/\zeta$, where $\zeta = \xi + i\eta$, for ζ'.

FIG.4

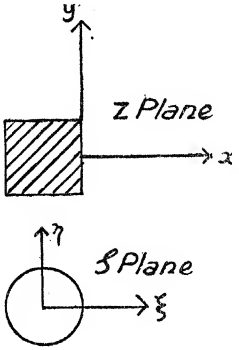
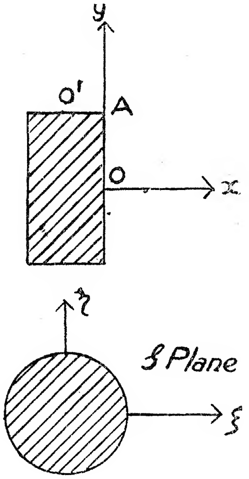


FIG.5



Thus $z = -\int_1^\zeta \sqrt{\left(\frac{1}{\zeta^4} - 2\cos 2\alpha \frac{1}{\zeta^2} + 1\right)}\, d\zeta$ converts the exterior of the rectangle in the z plane into the exterior of the circle $\xi^2 + \eta^2 = 1$ in the ζ plane, the circles at infinity in the two planes corresponding to each other, and for all large values of z or ζ, z being equal to $-\zeta$ (fig. 5).

If we now write $iw = \log \zeta$, where $w = u + iv$, v is the velocity at any point of the z plane. On substituting in the original equation we obtain

$$z = -i \int_0^w \sqrt{(e^{-4iw} - 2\cos 2\alpha \cdot e^{-2iw} + 1)}\, e^{iw}\, dw$$

as the equation connecting the position of a point in the z plane with the velocity at the point.

Introducing circular functions the equation becomes

$$z = \sqrt{2}\int_0^w \sqrt{(\cos 2\alpha - \cos 2w)}\, dw = 2\int_0^w \sqrt{(\sin^2\alpha - \sin^2 w)}\, dw.$$

On putting $\sin w = \sin \alpha \sin w'$* this becomes

$$z = 2\int_0^{w'} \frac{\sin^2 \alpha(1-\sin^2 w')}{\sqrt{(1-\sin^2 \alpha \sin^2 w')}}\, dw'$$

$$= 2\int_0^{w'} \sqrt{(1-\sin^2 \alpha \sin^2 w')}dw' - 2\int_0^{w'} \frac{\cos^2 \alpha}{\sqrt{(1-\sin^2 \alpha \sin^2 w')}}\, dw'.$$

From which we deduce

$$z = 2\mathrm{E}(w'\alpha) - 2\cos^2 \alpha \mathrm{F}(w'\alpha)$$

where F and E are the first and second elliptic integrals respectively.

At A the corner of the rectangle immediately to the right of the origin O in the z plane we have $w' = \frac{1}{2}\pi$ and $x_{\mathrm{A}} = 2\ \mathrm{E}_1(\alpha) - 2\cos^2\alpha\ .\ \mathrm{F}_1(\alpha)$ where F_1 and E_1 are the complete elliptic integrals.

The resistance of the half side OA of the rectangle is proportional to w_{A}, and since $\sin w_{\mathrm{A}} = \sin \alpha \sin \frac{1}{2}\pi = \sin \alpha$, $w_{\mathrm{A}} = \alpha$, and the resistance of the side OA is equal to $\alpha/2\pi$ of the resistance of the whole rectangle, or of that of a circular cylinder of radius 1, moving with the same speed.

When w exceeds α, the value of z becomes complex, its real part remaining $= x_{\mathrm{A}}$. If O′ is the middle point of the second side which starts at the corner A, then at O′, $w = \frac{1}{2}\pi$, $z_{\mathrm{O}'} = x_{\mathrm{A}} + iy_{\mathrm{O}'}$, and

$$y_{\mathrm{O}'} = 2\mathrm{E}(\tfrac{1}{2}\pi - \alpha) - 2\sin^2 \alpha\ .\ \mathrm{F}(\tfrac{1}{2}\pi - \alpha).$$

The resistance of the half side AO′ is equal to $(\frac{1}{2}\pi - \alpha)/2\pi$ of the resistance of the whole rectangle, or of that of a circular cylinder of radius 1, moving with the same speed.

7. The following Table gives the corresponding values of α, x_{A}, and $y_{\mathrm{O}'}$, that is of the half-sides of various rectangular prisms having the same resistance as a circular cylinder of radius unity moving with the same speed. A column giving the ratio of twice the length of the short to the sum of the lengths of long and short sides of the rectangle is given, to facilitate the choice of α in calculating the resistance in any given case. The last column contains the length of the perimeter of the rectangle, which may be compared with the perimeter $2\pi = 6{\cdot}28$ of the circular cylinder of the same resistance. In each case the resistance of the short half-side is equal to that part of the perimeter of the cylinder subtending an angle α at the centre, and that of the long half-side equal to that of the perimeter subtending the angle $\frac{1}{2}\pi - \alpha$.

For many purposes an approximate value of the total resistance of the prism is all that is required. The following relation may then be used. If

---

* On writing $\sin w = \sin a \operatorname{sn} \theta$ we get $z = 2\operatorname{gz}(\theta,\ a)$, where gz is the symmetrical elliptic function introduced by Glaisher, 'Quart. Journ. Math.,' vol. 20, p. 350 (1885).

α.	Short half-side.	Long half-side.	$\frac{\text{Short + short}}{\text{Long + short}}$.	Total perimeter.
0°	0·0	2·0	0·0	8·00*
5	0·0118	1·967	0·012	7·92
10	0·0478	1·890	0·050	7·75
15	0·106	1·782	0·112	7·55
20	0·187	1·651	0·203	7·35
25	0·287	1·503	0·321	7·16
30	0·406	1·344	0·464	7·00*
35	0·541	1·179	0·629	6·88
40	0·689	1·111	0·811	6·80
45	0·847	0·847	1·000	6·78*
Circular cylinder of same resistance..				6·28*

* These results have been verified experimentally by Mr. T. W. Blackaby, one of my students, by measuring the conductance of a layer of electrolyte 1 cm. deep between a strip, a rectangle, a square and a circle respectively, and an outer circle of circumference six or seven times that of the inner electrode.

L is the sum of the lengths of the long, and l that of the short sides of the rectangle, the circumference of a circular cylinder having the same resistance when moving with the same speed is given by

$$\text{equivalent circumference} = (\mathrm{L}+l)\{0{\cdot}785+0{\cdot}202\sqrt{[l/(\mathrm{L}+l)]}\}.$$

This relation is accurate to within 2 per cent.

8. *Case* 3.—A plane in motion parallel to itself in a viscous fluid has a fin or keel projecting into the fluid at right angles to its surface, its length being in the direction of motion. To find the effect of the keel on the resistance of the surface, the motion in the fluid being stream-line motion.

The Schwarzian transformation $z = \int\sqrt{[(a^2-w^2)/(b^2-w^2)]}\,dw$ converts the outline of the section of the surface and keel in the z plane, where $z = x+iy$, into the axis of u in the w plane, where $w = u+iv$, the middle point of the contour of the keel being taken as origin of co-ordinates in the z plane and becoming the origin of co-ordinates in the w plane, the outer corners of the keel becoming the points $\pm a$ and the inner corners the points $\pm b$ in the w plane (fig. 6).

Writing $w = a \operatorname{sn} \zeta$ the equation becomes

$$z = b(\int \mathrm{dn}^2\zeta d\zeta - k'^2\int d\zeta) = b[\mathrm{E}(\zeta, k) - k'^2\zeta]$$

where $k = a/b$, $k'^2 = 1-k^2$ and $\mathrm{E}(\zeta k)$ is the second elliptic integral.

If A is the outer corner of the projecting keel, OA is given by

$$x_{\mathrm{A}} = b\,[\mathrm{E}_1(k)-k'^2\mathrm{K}],$$

where $E_1(k)$ is the complete integral and K is the real quarter-period of ζ. The resistance to the motion of the strip of breadth OA is equal to that of a strip of breadth $a = kb$ of an infinite uniform plane moving with the same speed.

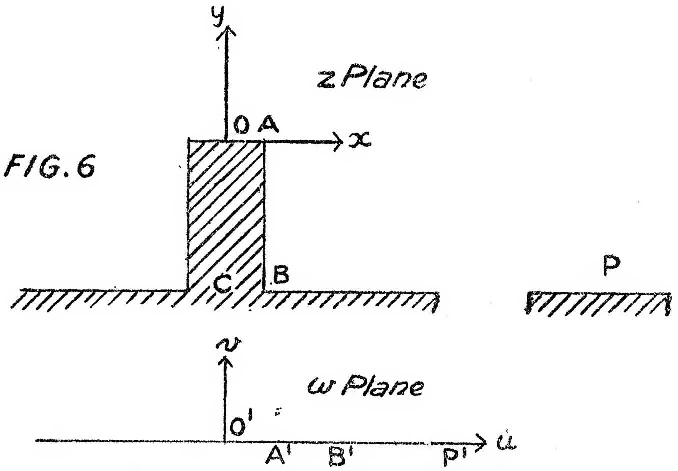


FIG. 6

For points in the plane surface beyond the inner corner B of the keel we have $w = a\,\mathrm{sn}\,(i\mathrm{K}' + \xi')$, $z = b\{\mathrm{E}(i\mathrm{K}' + \xi') - k'^2(i\mathrm{K}' + \xi')\}$.

These equations become on expansion

$$w = b/\mathrm{sn}\,\xi', \quad z = b\{\mathrm{E}(\xi', k) + \mathrm{cn}\,\xi'\,\mathrm{dn}\,\xi'/\mathrm{sn}\,\xi' - k'^2\xi' - i[\mathrm{E}_1(k') - k^2\mathrm{K}']\}.$$

At the point B, $w = b$, and $z = b\{\mathrm{E}_1(k) - k'^2\mathrm{K} - i(\mathrm{E}_1(k') - k^2\mathrm{K}')\}$.

Thus the resistance to the motion of the strips OA and AB is equal to that of a strip of breadth b of an infinite plane moving with the same speed.

When the point z is on the surface at a considerable distance from the keel, ξ is small and we may write

$$w = b/\xi'[1 - \tfrac{1}{6}(1 + k^2\xi'^2)] \quad \text{or} \quad \xi' = b/w\,.\,[1 + \tfrac{1}{6}(1 + k^2)\,.\,b^2/w^2]$$

and $$z = w + \{b^2/w\}[k^2 - \tfrac{1}{2}(1 + k^2)] - ib[\mathrm{E}_1(k') - k^2\mathrm{K}'] \quad \text{very nearly.}$$

Hence $x = w - b^2k'^2/2w$ or $w^2 = x^2 + b^2k'^2$, terms in $1/w^2$ being neglected.

Thus the resistance of the surface between the middle line of the keel and a parallel on the surface through P at a considerable distance x from the keel is equal to that of a strip of width $\sqrt{(x^2 + b^2k'^2)}$ of a plane surface moving with the same speed.

The ratio of the half width $\mathrm{OA} = b[\mathrm{E}(k) - k'^2\mathrm{K}]$ to the depth

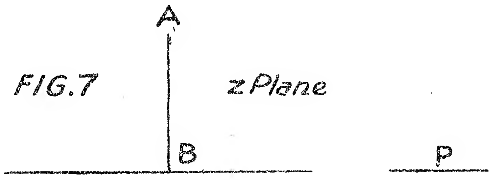
$$\mathrm{AB} = b[\mathrm{E}_1(k') - k^2\mathrm{K}']$$

determines k and k', then $b = \mathrm{AB}/[\mathrm{E}_1(k') - k^2\mathrm{K}']$ gives the other term in the resistance formula. It should be noticed that the accuracy of the approximate calculation above will decrease slightly as k approaches unity.

When k becomes small the keel becomes thin in proportion to its depth, and when k is zero it has become an infinitely thin strip projecting a distance b from the surface at right angles to it. Putting $k =$ zero in the general equation for z when $w > b$, it becomes

$$z = \sqrt{(w^2-b^2)}-ib,$$

as one would anticipate from the direct transformation.



Hence if P is a point some distance from the projecting strip (fig. 7)

$$x_p = \sqrt{(w^2-b^2)} \qquad y_p = -b.$$

Thus the resistance of one side of the projecting strip AB and the portion of the surface between the base B of the strip and the point P has the same resistance as a strip of width equal to AP of a plane surface moving with the same speed.

9. When the keel is not sufficiently thin to allow us to write $k = 0$ the accurate calculation of the resistance involves the use of tables of elliptic functions. Since in most practical cases an approximate value expressed in terms of simple functions will suffice, we proceed to find a simpler expression.

We have found that $\mathrm{OA} = b(\mathrm{E}-k'^2\mathrm{K})$, and $\mathrm{AB} = b(\mathrm{E}'-k^2\mathrm{K}')$.

Now $\mathrm{E}-k'^2\mathrm{K}$ is Glashier's G function* and can be expressed in terms of k by the series

$$\mathrm{G} = \frac{\pi}{4}\left(k^2+\frac{1^2}{2^2 \,.\, 2}k^4+\frac{1^2 \,.\, 3^2}{2^2 \,.\, 4^2 \,.\, 3}k^6+\ldots\right).$$

Similarly $\mathrm{E}'-k^2\mathrm{K}'$ is Glaisher's

$$\mathrm{G}' = \frac{\pi}{4}\left(k'^2+\frac{1^2}{2^2 \,.\, 2}k'^4+\frac{1^2 \,.\, 3^2}{2^2 \,.\, 4^2 \,.\, 3}k'^6+\ldots\right).$$

With the help of these series it may be shown that for small values of k'^2 the correcting term $b^2k'^2$ is proportional to $\mathrm{AB}^2+\mathrm{OA}\,.\,\mathrm{AB}$, but for values of k'^2 exceeding 0·4 and of OA/AB less than 1·5 it is not. A slight generalisation of this form of expression gives the correction to within 0·2 per cent., so long as OA/AB does not exceed 2, and to within 5 per cent. so long as it does not exceed 3. We shall take $b^2k'^2 = \mathrm{AB}^2 + 1{\cdot}77\,\mathrm{OA}^{0{\cdot}82}\mathrm{AB}^{1{\cdot}18}$ as a sufficiently close approximation to the correct value for our present purpose.

---

* Glaisher, 'Quart. Journ. Math.,' vol. 20, p. 320 (1885).

Thus the correction which has to be added to CP^2 (fig. 6) to give the square of the breadth of a strip of a plane surface of equal resistance to OA, AB, and BP, when moving with the same speed, is $\mathrm{AB}^2 + 1{\cdot}77\,\mathrm{OA}^{0{\cdot}82}\mathrm{AB}^{1{\cdot}18}$, that is the breadth of the strip of a plane surface equal in resistance to the strips OA, AB and BP is $\sqrt{[(\mathrm{OA}+\mathrm{BP})^2 + \mathrm{AB}^2 + 1{\cdot}77\,\mathrm{OA}^{0{\cdot}82}\,\mathrm{AB}^{1{\cdot}18}]}$ for all cases in which OA does not exceed twice AB.

10. The cases considered in the previous paragraphs show that for motions of very elongated bodies parallel to their lengths through fluids in which stream-line motion is produced by the movement, the influence of the form and magnitude of the section of the moving body on the viscous resistance to the motion is considerable. Values of the resistances per unit area of contact with the fluid and the total resistances are given for the following cases:—

1. A circular pipe with a concentric core, the fluid moving through the annular space between.
2. A pipe of elliptical section through which fluid is moving.
3. A long body of elliptical section in motion through a fluid. The limiting cases in which the section becomes a circle and a straight line respectively are considered.
4. A long body of rectangular section in motion through a fluid. Square and straight line sections are considered.
5. A long wide surface from which a keel with its length along the line of motion projects. The ratios of width to depth of keel cover a considerable range.

In all cases the law of resistance may be expressed in a simple form, and it is desirable that measurements of the resistances in the corresponding cases in which the motion in the fluid becomes turbulent should be made* in order if possible to carry over these laws to turbulent motion or to provide any corrections to them which the new circumstances entail.

---

* Experiments in this direction are already being made by Mr. N. A. V. Tonnstein, B.Sc., at the East London College, and are leading to interesting results.